M321/W

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THE OPEN UNIVERSITY

Third Level Course Examination 1976

PARTIAL DIFFERENTIAL EQUATIONS OF APPLIED MATHEMATICS

Wednesday, 3rd November, 1976 **10.00 a.m. – 1.00 p.m.**

Time allowed: 3 hours

You should attempt **ALL** the questions from Section A, and **NOT MORE THAN FOUR** questions from Section B. Section A carries about 40% of the marks.

You may answer questions in any order, writing your answers in the answer books provided. **Use a separate answer book for each section.** At the end of the examination, remember to write your name, student number and examination number on the answer books – **failure to do so will mean that your papers cannot be identified.**

SECTION A
(about 40 marks)

Attempt ALL the questions in this section.
Not all the questions in this section carry equal marks.
Write your answers in one of the answer books provided. Do NOT use the same answer book for this section as for Section B.

Question 1

Given that $x = r\cos\theta$ and $y = r\sin\theta$, and that $u(x, y)$ is a function on the (x, y) plane, express

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

in terms of r, θ, $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial \theta}$.

Question 2

A function f with domain R satisfies

$$f(x) = x^2$$

for $0 \leqslant x \leqslant 2$, and is odd about $x = 0$ and even about $x = 2$. Sketch its graph for $-6 \leqslant x \leqslant 6$ and write down (a) the period of f and (b) the formula giving $f(x)$ for $4 \leqslant x \leqslant 6$.

Question 3

Match the following three differential equations to the most appropriate boundary conditions and domains from the three alternatives supplied, so as to give three properly posed problems.

A $\quad \dfrac{\partial^2 u(x, y)}{\partial x^2} + 2\dfrac{\partial^2 u(x, y)}{\partial y^2} = 0$,

B $\quad \dfrac{\partial^2 u(x, y)}{\partial x^2} - 2\dfrac{\partial^2 u(x, y)}{\partial y^2} = 0$,

C $\quad \dfrac{\partial^2 u(x, y)}{\partial x^2} - 2\dfrac{\partial u(x, y)}{\partial y} = 0$,

a $\quad \left\{\begin{array}{l} u(0, y) = y,\ u(x, 0) = 0 \\ \dfrac{\partial u}{\partial x}(1, y) = 0 \end{array}\right\} \ (0 < x < 1, \quad 0 < y)$,

b $\quad \left\{\begin{array}{l} u(0, y) = y, \quad u(x, 0) = 0 \\ \dfrac{\partial u}{\partial x}(1, y) = 0, \dfrac{\partial u}{\partial y}(x, 0) = 1 \end{array}\right\} \ (0 < x < 1, \quad 0 < y)$,

c $\quad \left\{\begin{array}{l} u(0, y) = y, \quad u(x, 0) = 0 \\ \dfrac{\partial u}{\partial x}(1, y) = 0, u(x, 1) = 1 \end{array}\right\} \ (0 < x < 1, \quad 0 < y < 1)$.

Question 4

The Fourier series

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi}{3}\right)\cos n\pi x$$

converges (pointwise) to the sum $\frac{2}{3}$ when $-\frac{2}{3} < x < \frac{2}{3}$ and to the sum $-\frac{4}{3}$ when $\frac{2}{3} < x < 1$. To what sum does it converge (a) when $x = \frac{2}{3}$, (b) when $x = -\frac{2}{3}$? Give the reason for your answer.

Considered as a series of functions with domain $[-1, 1]$, does the series converge uniformly? Give the reason for your answer.

Question 5 What is meant by *convergence* of a finite difference approximation to a differential equation?

A function $U(x, t)$, which satisfies the one-dimensional heat equation $\dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2}$, is to be approximated over the mesh $x_i = ih$, $t_j = jk$ with $k = \frac{1}{2}h^2$, using the following finite difference approximation to the heat equation:

$$u_{i,j+1} = \tfrac{1}{2}[u_{i-1,j} + u_{i+1,j}] .$$

Given that the true solution U satisfies

$$U_{i,j+1} = \tfrac{1}{2}[U_{i-1,j} + U_{i+1,j}] + O(k^2)$$

where $U_{i,j}$ means $U(x_i, t_j)$, deduce that the approximation method is convergent.

Question 6 Laplace's equation in two-dimensional polar coordinates (r, θ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad (r > 0,\ 0 \leqslant \theta \leqslant 2\pi) .$$

The method of separation of variables yields as a solution

$$u = (Ar^k + Br^{-k})(C\cos k\theta + D\sin k\theta)$$

where A, B, C, D, k are constants and $k > 0$. For a solution of this type what conditions on these constants are implied (a) by the condition that u must be single-valued, and bounded in the neighbourhood of the origin, (b) by the condition that u must be single-valued and bounded for arbitrarily large r?

Question 7 This question is about the system of equations

$$\begin{aligned} 10x_1 - x_2 \phantom{{}- x_3} &= 1 \\ -x_1 + 10x_2 - x_3 &= 2 \\ - x_2 + 10x_3 &= 3 . \end{aligned}$$

(a) Starting with the approximation $x_1 = x_2 = x_3 = 0$, calculate the next approximation using the Jacobi method.

(b) Same as (a), but using the Gauss–Seidel method.

(c) Given that the eigenvalues of the Jacobi iteration matrix

$$\begin{bmatrix} 0 & \frac{1}{10} & 0 \\ \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & \frac{1}{10} & 0 \end{bmatrix}$$

are 0, $\frac{1}{10}\sqrt{2}$, $-\frac{1}{10}\sqrt{2}$, give an estimate of the factor by which the accuracy of the approximate solution is increased at each application of the Gauss–Seidel method.

Question 8 By comparing with a suitable constant-coefficient equation, obtain a positive lower bound for the lowest eigenvalue λ of the problem

$$\frac{d^2u(x)}{dx^2} - xu(x) + \lambda u(x) = 0 \qquad (1 < x < 2)$$

$$u(1) = u(2) = 0 .$$

SECTION B (about 60 marks)

Attempt at most FOUR questions from this part.
Write your answers in one of the answer books provided. Do NOT use the same answer book for this section as for Section A.

Question 9 (i) Explain what is meant by the word *characteristics* in connection with hyperbolic equations.

(ii) Find the characteristics of the equation

$$9\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial x\,\partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0 \qquad (x, y \in R) .$$

(iii) Hence, or otherwise, find a change of independent variables which reduces this system to the standard form $\dfrac{\partial^2 u}{\partial \xi\,\partial \eta} = 0$ (it is not necessary to carry out the actual reduction).

(iv) If the values of u and $\dfrac{\partial u}{\partial x}$ are specified along the straight line

$$y = mx + c ,$$

for what values of the constants m and c would it be impossible to find a unique solution? Give the reason for your answer.

Question 10 (i) Classify the equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} + 4x\frac{\partial^2 u(x,t)}{\partial x\,\partial t} + 4x^2\frac{\partial^2 u(x,t)}{\partial t^2} = 0 \qquad (x, t \in R)$$

as elliptic, hyperbolic, or parabolic.

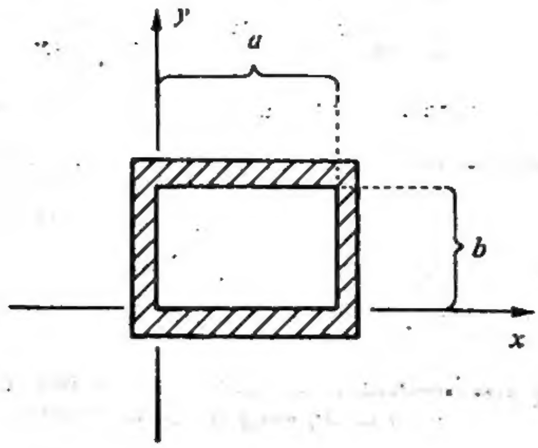
(ii) How many characteristics of this equation pass through a typical point in the (x, t) plane?

(iii) Find the characteristics.

(iv) Find a coordinate system in which the equation takes its standard form.

(v) Carry out the transformation to standard form.

Question 11



The diagram shows the cross-section of a pipe which is uniform in the direction perpendicular to the paper. The inner contour is rectangular. Write down a differential equation and boundary conditions which model the steady flow of a viscous fluid along this pipe, defining any new symbols introduced (apart from those in the diagram) and stating what assumptions you are making about the direction and about the space and time dependence of the velocity vector.

Use the second extremum principle for viscous flow, with a trial vector field of the form $\mathbf{v}^* = (\alpha x, \beta y)$ where α, β are constants, to obtain an upper bound on the volume of fluid per unit time delivered by the pipe. What values of α, β give the best such bound?

Let $\mathbf{v}_0^*$ denote the trial vector field (not necessarily of the form $(\alpha x, \beta y)$) which gives the best possible upper bound in the second extremum principle. Express the components of $\mathbf{v}_0^*$ in terms of the velocity of the fluid in the pipe and its derivatives.

Question 12 Describe one explicit and one implicit finite-difference scheme for solving the problem

$$\frac{\partial U(x,t)}{\partial t} = 2\frac{\partial^2 U(x,t)}{\partial x^2} + \frac{\partial U(x,t)}{\partial x} \qquad (0 < x < 1,\ t > 0)$$

$$\left.\begin{aligned} U(0,t) &= 0 \\ \frac{\partial U(1,t)}{\partial x} &= 0 \end{aligned}\right\} \qquad (t \geqslant 0)$$

$$U(x,0) = x^2 - 2x \qquad (0 \leqslant x \leqslant 1)\ .$$

Compare and contrast your two schemes with respect to stability, accuracy, and ease of computation, including detailed reasons for any assertions you make.

Question 13 (i) Bring the equation

$$\frac{d^2u(x)}{dx^2} + \frac{du(x)}{dx} - 2u(x) = f(x) \qquad (0 < x < 1)$$

to self-adjoint form.

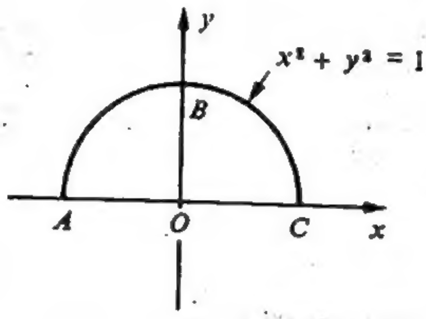
(ii) Find the Green's function $G(x, \xi)$ for the problem of solving the above equation with the boundary conditions

$$u(0) = u(1) = 0 \; .$$

(iii) Verify, if you have not already done so as part of the calculation of $G(x, \xi)$, that $G(x, \xi)$ is continuous and has the correct discontinuity in its derivative at $x = \xi$.

(iv) Indicate briefly, and without doing any calculations, how you could use this Green's function to solve the given problem.

Question 14



(i) Let D denote the semicircular region of the (x, y) plane shown in the diagram. If u is a function defined on D, satisfying $\frac{\partial u}{\partial y} = 0$ when $y = 0$, show that the corresponding function for polar coordinates (which we also denote by u) satisfies

$$\frac{\partial u(r, 0)}{\partial \theta} = \frac{\partial u(r, \pi)}{\partial \theta} = 0 \qquad (0 < r < 1) \; .$$

(ii) Reformulate the following problem in polar coordinates:

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F \qquad ((x, y) \in D) \\ &u = 0 \qquad \text{on the boundary curve } ABC \\ &\frac{\partial u}{\partial n} = 0 \qquad \text{on the boundary } AOC \end{aligned}\right\} \qquad (1)$$

where F is an arbitrary continuous function on D.

(iii) The Green's function for the problem (1), expressed in polar coordinates, has the form

$$G(r, \theta; \rho, \phi) = a \ln\left\{\frac{r^2 - 2r\rho\cos(\theta - \phi) + \rho^2}{r^2\rho^2 - 2r\rho\cos(\theta - \phi) + 1}\right\}$$
$$+ \, b \ln\left\{\frac{r^2 - 2r\rho\cos(\theta + \phi) + \rho^2}{r^2\rho^2 - 2r\rho\cos(\theta + \phi) + 1}\right\} \; .$$

Evaluate the constants a and b.

Question 15 A Dirichlet problem for the equation

$$\frac{\partial^2 U(x, y)}{\partial x^2} + \frac{\partial^2 U(x, y)}{\partial y^2} + \frac{\partial U(x, y)}{\partial x} = 0$$

is to be solved approximately, using a square mesh of side h and the finite-difference scheme

$$4u_{i,j} - (1 + \tfrac{1}{2}h)u_{i+1,j} - (1 - \tfrac{1}{2}h)u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0$$

where $u_{i,j}$ is the approximation to $U(x_i, y_j)$ and $x_i = x_0 + ih$, $y_j = y_0 + jh$.

(i) For what range of values of h does the resulting system of simultaneous equations satisfy a maximum principle?

(ii) Given that h satisfies the condition obtained in (i), use the maximum principle to prove that the system of equations has a unique solution.

(iii) Prove that the finite difference scheme is a consistent approximation to the differential equation.

Question 16 Use the method of separation of variables to obtain the general solution in series for the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad (0 < r < a, 0 < z < b, t > 0)$$

where $u = u(r, z, t)$ and a, b, c are positive constants, subject to the boundary conditions:

$u(r, z, t)$ bounded in the neighbourhood of $r = 0$,
$u(r, z, t) = 0$ if $r = a$, or if $z = 0$ or $z = b$, or if $t = 0$.

Express your answer in terms of the zero-order Bessel function J_0 and its zeros $j_n^{(0)}$.

Question 17 It is proposed to use the finite-difference scheme given by the approximations

$$\frac{\partial^2 u_{i,j}}{\partial t^2} \simeq \frac{1}{k^2}\delta_t^2 u_{i,j}$$

$$\frac{\partial^2 u_{i,j}}{\partial x^2} \simeq \frac{1}{h^2}[a\delta_x^2 u_{i,j+1} + (1 - 2a)\delta_x^2 u_{i,j} + a\delta_x^2 u_{i,j-1}],$$

where a is a constant, to calculate an approximate solution of

$$4\frac{\partial^2 U(x, t)}{\partial t^2} = [U(x, t)]^2 \frac{\partial^2 U(x, t)}{\partial x^2}$$

over the mesh $x_i = x_0 + ih$, $t_j = t_0 + jk$, where $u_{i,j}$ stands for the approximation to $U(x_i, t_j)$.

(i) Write out the finite difference scheme.

(ii) What does the von Neumann method indicate about the stability of this scheme when $a = \frac{1}{4}$?